

# INNER IDEALS, COMPACT TRIPOTENTS AND ČEBYŠEV SUBTRIPLES OF $JB^*$ -TRIPLES AND $C^*$ -ALGEBRAS

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**ABSTRACT.** The aim of this note is to study Čebyšev  $JB^*$ -subtriples of general  $JB^*$ -triples. It is established that if  $F$  is a non-zero Čebyšev  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ , then exactly one of the following statements holds:

- (a)  $F$  is a rank one  $JBW^*$ -triple with  $\dim(F) \geq 2$  (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover,  $F$  may be a closed subspace of arbitrary dimension and  $E$  may have arbitrary rank;
- (b)  $F = \mathbb{C}e$ , where  $e$  is a complete tripotent in  $E$ ;
- (c)  $E$  and  $F$  are rank two  $JBW^*$ -triples, but  $F$  may have arbitrary dimension;
- (d)  $F$  has rank greater or equal than three and  $E = F$ .

## 1. INTRODUCTION

It is known that certain problems on operator algebras are more feasible when the algebra under study is a von Neumann algebra (i.e. a  $C^*$ -algebra which is also a dual Banach space). For example, A.G. Robertson gave in [35] a complete description of one-dimensional Čebyšev subspaces, and the finite dimensional Čebyšev hermitian subalgebras with dimension bigger than 1 of a general von Neumann algebra. Concretely, for a non-zero element  $x$  in a von Neumann algebra  $M$ , subspace  $\mathbb{C}x$  is a Čebyšev subspace of  $M$  if and only if there is a projection  $p$  in the center of  $M$  such that  $px$  is left invertible in  $pM$  and  $(1 - p)x$  is right invertible in  $(1 - p)M$  (cf. [35, Theorem 1]). A finite dimensional  $*$ -subalgebra  $N$  of an infinite dimensional von Neumann algebra  $M$  with  $\dim(N) > 1$  never is a Čebyšev subspace of  $M$  (see [35, Theorem 6]).

Two years were needed to relax the assumptions concerning duality, to finally obtain valid answers for Čebyšev subspaces and subalgebras of a general  $C^*$ -algebra. A.G. Robertson and D. Yost proved in [36, Corollary 1.4] that an infinite dimensional  $C^*$ -algebra  $A$  admits a finite dimensional  $*$ -subalgebra  $B$  which is also a Čebyšev in  $A$  if and only if  $A$  is unital and  $B = \mathbb{C}1$ . The results proved by Robertson and Yost were complemented by G.K. Pedersen, who shows that if  $A$

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is a  $C^*$ -algebra without unit and  $B$  is a Čebyšëv  $C^*$ -subalgebra of  $A$ , then  $A = B$  (compare [34, Theorem 4]).

We recall that a subspace  $V$  of a Banach space  $X$  is called a *Čebyšëv* (*Cheby-shev*) *subspace* of  $X$  if for each  $x \in X$  there exists a unique point  $\pi_V(x) \in V$  such that  $\text{dist}(x, V) = \|x - \pi_V(x)\|$ . Throughout this note the symbol  $\pi_V(x)$  will denote the best approximation of an element  $x$  in  $X$  in a Čebyšëv subspace  $V$  of  $X$ . For more information on Čebyšëv and best approximation theory we refer to the monograph [37].

Similar benefits to those obtained working with von Neumann algebras reappear when studying Čebyšëv subspaces which Ternary Rings of Operators (TRO's) of a given von Neumann algebra, or when exploring Čebyšëv  $JBW^*$ -subtriples of a given  $JBW^*$ -triple (see section 2 for definitions). In a previous paper, we establish the following description of Čebyšëv  $JBW^*$ -subtriples of a  $JBW^*$ -triple.

**Theorem 1.** [26, Theorem 13] *Let  $N$  be a non-zero Čebyšëv  $JBW^*$ -subtriple of a  $JBW^*$ -triple  $M$ . Then exactly one of the following statements holds:*

- (a)  *$N$  is a rank one  $JBW^*$ -triple with  $\dim(N) \geq 2$  (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover,  $N$  may be a closed subspace of arbitrary dimension and  $M$  may have arbitrary rank;*
- (b)  *$N = \mathbb{C}e$ , where  $e$  is a complete tripotent in  $M$ ;*
- (c)  *$N$  and  $M$  have rank two, but  $N$  may have arbitrary dimension;*
- (d)  *$N$  has rank greater or equal than three and  $N = M$ .* □

The question whether in the above theorem  $JBW^*$ -triples and subtriples can be replaced with  $JB^*$ -triples and subtriples remains as an open problem. The techniques employed in [26] rely heavily on the rich geometric properties of  $JBW^*$ -triples. In this note we study this problem in the more general setting of  $JB^*$ -triples. We combine here new arguments involving inner ideals and compact tripotents in the bidual of a  $JB^*$ -triple. The main result of this note shows that the conclusion of the above Theorem 1 also holds when  $N$  is a  $JB^*$ -subtriple of a general  $JB^*$ -triple  $M$  (see Theorem 15).

Among the new results proved in this note we also establish that a Čebyšëv  $C^*$ -subalgebra  $B$  (respectively, a Čebyšëv  $JB^*$ -subtriple) of a  $C^*$ -algebra  $A$  with  $\text{rank}(B) \geq 3$  coincides with the whole  $A$  (see Corollary 13).

## 2. PRELIMINARIES

The multiple attempts to understand a Riemann mapping theorem type for complex Banach spaces of dimension bigger or equal than 2, led some mathematicians to the study of bounded symmetric domains (compare [10, 33, 23, 24] and [29]). The definite answer was given by W. Kaup, who showed the existence of a set of algebraic-geometric-analytic axioms which determine a class of complex Banach spaces, the class of  $JB^*$ -triples, whose open unit balls are bounded symmetric domains, and every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a  $JB^*$ -triple; in this

way, the category of all bounded symmetric domains with base point is equivalent to the category of  $JB^*$ -triples.

A  $JB^*$ -triple is a complex Banach space  $E$  with a continuous triple product  $(a, b, c) \mapsto \{a, b, c\}$ , which is bilinear and symmetric in the external variables and conjugate linear in the middle one and satisfies:

(a) (Jordan identity)

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all  $x, y, a, b, c \in E$ , where  $L(x, y) : E \rightarrow E$  is the linear mapping given by  $L(x, y)z = \{x, y, z\}$ ;

(b) For each  $x \in E$ , the operator  $L(x, x)$  is hermitian with non-negative spectrum;

(c)  $\|\{x, x, x\}\| = \|x\|^3$  for all  $x \in E$ .

Given an element  $a$  in a  $JB^*$ -triple  $E$ , the symbol  $Q(a)$  will denote the conjugate linear map on  $E$  defined by  $Q(a)(x) := \{a, x, a\}$ .

The class of  $JB^*$ -triples includes all  $C^*$ -algebras when the latter are equipped with the triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a). \quad (2.1)$$

The space  $B(H, K)$  of all bounded linear operators between complex Hilbert spaces, although rarely is a  $C^*$ -algebra, is a  $JB^*$ -triple with the product defined in (2.1). In particular, every complex Hilbert space is a  $JB^*$ -triple. Thus, the class of  $JB^*$ -triples is strictly wider than the class of  $C^*$ -algebras.

A  $JBW^*$ -triple is a  $JB^*$ -triple which is also a dual Banach space (with a unique isometric predual [1]). The triple product of every  $JBW^*$ -triple is separately weak\* continuous (cf. [1]). The second dual,  $E^{**}$ , of a  $JB^*$ -triple,  $E$ , is a  $JBW^*$ -triple with a certain triple product extending the product of  $E$  (cf. [12]).

The study of Čebyšev  $JB^*$ -subtriples of a given  $JB^*$ -triple requires some other examples which are very well known. A Cartan factor of type 1 is a  $JB^*$ -triple which coincides with the Banach space  $B(H, K)$  of bounded linear operators between two complex Hilbert spaces,  $H$  and  $K$ , where the triple product is defined by (2.1). Cartan factors of types 2 and 3 are  $JB^*$ -triples which can be identified the subtriples of  $B(H)$  defined by  $II^{\mathbb{C}} = \{x \in B(H) : x = -jx^*j\}$  and  $III^{\mathbb{C}} = \{x \in B(H) : x = jx^*j\}$ , respectively, where  $j$  is a conjugation on  $H$ . A Cartan factor of type 4 or  $IV$  is a spin factor, that is, a complex Hilbert space provided with a conjugation  $x \mapsto \bar{x}$ , where the triple product and the norm are defined by

$$\{x, y, z\} = \langle x/y \rangle z + \langle z/y \rangle x - \langle x/\bar{z} \rangle \bar{y},$$

and  $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - |\langle x/\bar{x} \rangle|^2}$ , respectively. The Cartan factors of types 5 and 6 consist of finite dimensional spaces of matrices over the eight dimensional complex Cayley division algebra  $\mathbb{O}$ ; the type  $VI$  is the space of all hermitian  $3 \times 3$  matrices over  $\mathbb{O}$ , while the type  $V$  is the subtriple of  $1 \times 2$  matrices with entries in  $\mathbb{O}$  (compare [33], [21], and [11, §2.5]).

Let  $E$  be a  $JB^*$ -triple. An element  $e \in E$  is called a *tripotent* if  $\{e, e, e\} = e$ . For each tripotent  $e \in E$ , the eigenspaces of the operator  $L(e, e)$  induce a

decomposition (called *Peirce decomposition*) of  $E$  in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for  $i = 0, 1, 2$ ,  $E_i(e) = \{x \in E : L(e, e)(x) = \frac{i}{2}x\}$  (compare [33, Theorem 25]). The natural projections of  $E$  onto  $E_i(e)$  will be denoted by  $P_i(e)$ . It is known that this decomposition satisfies the following multiplication rules:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if  $i - j + k \in \{0, 1, 2\}$  and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

A tripotent  $e$  in  $E$  is called *complete* (respectively, *minimal*) if the equality  $E_0(e) = 0$  (respectively,  $E_2(e) = \mathbb{C}e \neq \{0\}$ ) holds.

The connections between JB\*-triples and JB\*-algebras are very deep. Every JB\*-algebra is a JB\*-triple under the triple product defined

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \quad (2.2)$$

The Peirce space  $E_2(e)$  is a JB\*-algebra with product  $x \circ_e y := \{x, e, y\}$  and involution  $x^{\sharp_e} := \{e, x, e\}$ .

Let  $a$  be an element in a JB\*-triple  $E$ . It is known that the JB\*-subtriple,  $E_a$ , generated by  $a$ , identifies with some  $C_0(L_a)$  where  $\|a\| \in L_a \subseteq [0, \|a\|]$  with  $L_a \cup \{0\}$  compact (cf. [29, 1.15]). Moreover, there exists a triple isomorphism  $\Psi : E_a \rightarrow C_0(L_a)$  such that  $\Psi(a)(t) = t$ . Consequently, the symbol  $a^{\frac{1}{3}}$  will stand for the unique element in  $E_a$  satisfying  $\{a^{\frac{1}{3}}, a^{\frac{1}{3}}, a^{\frac{1}{3}}\} = a$ .

When  $a$  is an element in a JBW\*-triple  $M$ , the sequence  $(a^{\frac{1}{2n-1}})$  converges in the weak\*-topology of  $M$  to a tripotent, denoted by  $r(a)$ , called the *range tripotent* of  $a$ . The tripotent  $r(a)$  is the smallest tripotent  $e \in M$  satisfying that  $a$  is positive in the JBW\*-algebra  $M_2(e)$  (see [13, page 322]). Clearly, the range tripotent  $r(a)$  can be identified with the characteristic function  $\chi_{(0, \|a\|] \cap L_a} \in C_0(L_a)^{**}$  (see [6, beginning of §2]). If we define  $a^{[1]} = a$ ,  $a^{[3]} = \{a, a, a\}$  and  $a^{[2n+1]} = \{a, a, a^{[2n-1]}\}$ , for every  $n \geq 1$ , it is known that the sequence  $(a^{[2n-1]})$  converges in the weak\* topology of  $M$  to a tripotent (called the *support tripotent* of  $a$ )  $s(a)$  in  $E^{**}$ , which satisfies  $s(a) \leq a \leq r(a)$  in the JBW\*-algebra  $M_2(r(a))$  (compare [13, Lemma 3.3]).

We recall that two elements  $a, b$  in a JB\*-triple  $E$  are *orthogonal* (written as  $a \perp b$ ) if  $L(a, b) = 0$  (see [7, Lemma 1] for several equivalent reformulations). Given a subset  $M \subseteq E$ , we write  $M_E^\perp$  (or simply  $M^\perp$ ) for the (orthogonal) annihilator of  $M$  defined by  $M_E^\perp = \{y \in E : y \perp x, \forall x \in M\}$ . If  $e \in E$  is a tripotent, then  $\{e\}_E^\perp = E_0(e)$ , and  $\{a\}_E^\perp = (E^{**})_0(r(a)) \cap E$ , for every  $a \in E$  (cf. [8, Lemma 3.2]).

Given a tripotent  $e \in E$ , we know from Lemma 1.3(a) in [20] that

$$\|x_2 + x_0\| = \max\{\|x_2\|, \|x_0\|\},$$

for every  $x_2 \in E_2(e)$  and every  $x_0 \in E_0(e)$ . This geometric property with the equivalent reformulations of orthogonality given in [7, Lemma 1] we deduce that

$$\|a + b\| = \max\{\|a\|, \|b\|\}, \quad (2.3)$$

whenever  $a$  and  $b$  are orthogonal elements in a JB\*-triple. A subset  $S \subseteq E$  is said to be *orthogonal* if  $0 \notin S$  and  $x \perp y$  for every  $x \neq y$  in  $S$ . The minimal cardinal number  $r$  satisfying  $\text{card}(S) \leq r$  for every orthogonal subset  $S \subseteq E$  is called the *rank* of  $E$  (and will be denoted by  $r(E)$ ). Given a tripotent  $e \in E$ , the rank of the Peirce-2 subspace  $E_2(e)$  will be called the rank of  $e$ .

We shall also make use of a natural partial order defined on the set of tripotents (see Corollary 1.7 and comments preceding it in [20]). Given two tripotents  $e, u$  in a JB\*-triple  $E$ , we say that  $e \leq u$  if  $u - e$  is a tripotent in  $E$  with  $u - e \perp e$ .

We finally, recall that an element  $x$  in a JB\*-triple  $E$  is called *Brown-Pedersen quasi-invertible* (BP quasi-invertible for short) if there exists  $y \in E$  such that  $B(x, y) = 0$  (cf. [27]), where  $B(x, y)$  denotes the Bergmann operator  $B(x, y) = I_E - 2L(x, y) + Q(x)Q(y)$ . Theorems 6 and 11 in [27] prove that an element  $x$  in  $E$  is Brown-Pedersen quasi-invertible if, and only if,  $x$  is von Neumann regular in the sense of [15, 30, 9] and its range tripotent is an extreme point of the closed unit ball of  $E$ , equivalently, there exists a complete tripotent  $v \in E$  such that  $x$  is positive and invertible in  $E_2(v)$ . In particular, every extreme point of the closed unit ball of  $E$  is BP quasi-invertible.

### 3. ČEBYŠEV SUBTRIPLES OF JB\*-TRIPLES

The following auxiliary results were established in [26, §3]

**Proposition 2.** [26, Propositions 9 and 10] *Let  $F$  be a Čebyšev JB\*-subtriple of a JB\*-triple  $E$ . Suppose  $e$  is a non-zero tripotent in  $F$ . Then the following statements hold:*

- (a)  $E_0(e) = F_0(e)$ , and consequently, every complete tripotent in  $F$  is complete in  $E$ .
- (b) If  $F_0(e) = \{e\}_F^\perp \neq 0$ , then  $E_2(e) = F_2(e)$ . □

We continue, in this paper, our study on Čebyšev subtriples of general JB\*-triples.

We recall that orthogonal elements in  $E$  are always  $M$ -orthogonal, i.e.  $a \perp b$  in  $E$  implies that  $\|\alpha a + \beta b\| = \max\{\|\alpha a\|, \|\beta b\|\}$ , for all  $\alpha, \beta \in \mathbb{C}$  (see, for example, [7, Lemma 1] and [20, Lemma 1.3]).

**Proposition 3.** *Let  $F$  be a Čebyšev JB\*-subtriple of a JB\*-triple  $E$ . For each  $a$  in  $F$  which is not Brown-Pedersen quasi-invertible the norm closure of the space  $Q(a)(E)$  is contained in  $F$ .*

*Proof.* Since  $a$  is in  $F \setminus F_q^{-1}$  we have two possibilities either  $a$  is not von Neumann regular or  $a$  is von Neumann regular and its range tripotent is not an extreme point of the closed unit ball of  $F$ . We deal with each case separately. We can assume that  $\|a\| = 1$ .

Suppose first that  $a$  is not von Neumann regular. Then 0 is a non-isolated point in the triple spectrum  $L_a$  of  $a$ . We know that in this case,  $0, 1 \in L_a \subseteq [0, 1]$ , with  $L_a$  compact. We further know that, if  $F_a$  denotes the JB\*-subtriple of  $F$  generated by  $a$ , then there exists a triple isomorphism  $\Psi$  from  $F_a$  onto  $C_0(L_a)$ , satisfying  $\Psi(a)(t) = t$  ( $t \in L_a$ ), where  $C_0(L_a)$  denotes the commutative C\*-algebra of all continuous functions on  $L_a$  vanishing at zero (cf. [29, Lemma 1.14] and [30, §3]).

Fix a natural  $n$ , and define the following functions in  $F_a$ , defined as elements in  $C_0(L_a)$ ,

$$b_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{3n}]; \\ \text{affine}, & \text{if } t \in [\frac{1}{3n}, \frac{1}{2n}]; \\ 1, & \text{if } t \in [\frac{1}{2n}, 1] \end{cases} \quad a_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2n}]; \\ \text{affine}, & \text{if } t \in [\frac{1}{2n}, \frac{1}{n}]; \\ t, & \text{if } t \in [\frac{1}{n}, 1] \end{cases}$$

and

$$z_n(t) := \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } t = \frac{1}{5n}; \\ \text{affine}, & \text{otherwise}; \\ 0, & \text{if } t \in [\frac{1}{4n}, 1]. \end{cases}$$

Clearly  $\|a_n\| = \|z_n\| = \|b_n\| = 1$ ,  $\|a - a_n\| = \frac{1}{2n}$ ,  $\{b_n, a_n, b_n\} = a_n$ . and  $z_n \perp a_n, b_n$ , for every  $n \in \mathbb{N}$ . We claim that

$$\{a_n, E, a_n\} \subseteq F, \quad (3.1)$$

for every natural  $n$ . Suppose, on the contrary, that there exists an element  $w \in \{a_n, E, a_n\} \setminus F$ . Since in  $F_a^{**}$  (and hence in  $F^{**}$ )  $b_n = r(a_n) + (b_n - r(a_n))$ , where  $r(a_n) \perp (b_n - r(a_n))$  and  $r(a_n)$  is the range tripotent of  $a_n$  in  $F^{**}$ , and  $w \in \{a_n, E, a_n\}$ , we can easily see that  $Q(b_n)^2(w) = w$ . The element  $\pi_F(w)$  lies in  $F$ , and thus  $Q(b_n)^2(\pi_F(w)) \in F$ . We observe that

$$\|w - Q(b_n)^2(\pi_F(w))\| = \|Q(b_n)^2(w - \pi_F(w))\| \leq \|w - \pi_F(w)\| = \text{dist}(w, F).$$

The uniqueness of the best approximation of  $w$  in  $F$  implies that  $\pi_F(w) = Q(b_n)^2(\pi_F(w))$ , equality which implies that  $z_n \perp Q(b_n)^2(\pi_F(w)) = \pi_F(w)$ . We therefore have  $z_n \perp w + Q(b_n)^2(\pi_F(w))$  (because  $w = Q(b_n)^2(w) \perp z_n$ ). For each  $\lambda \in \mathbb{C}$ , the element  $\lambda z_n + \pi_F(w)$  belongs to  $F$ , and since orthogonal elements are  $M$ -orthogonal, we have

$$\begin{aligned} \|\lambda z_n + \pi_F(w)\| &= \|-\lambda z_n + (w - \pi_F(w))\| \\ &= \max\{\|w - \pi_F(w)\|, \|\lambda z_n\|\} = \text{dist}(w, F), \end{aligned}$$

for every  $|\lambda| \leq \|w - \pi_F(w)\| = \text{dist}(w, F)$ , which contradicts the uniqueness of  $\pi_F(w)$ . This proves the claim.

Now, since  $(a_n) \rightarrow a$  in norm, the triple product of  $E$  is norm continuous, and  $F$  is norm closed, we deduce from (3.1) that  $\{a_n, E, a_n\} \subseteq F$ , and hence  $\overline{\{a_n, E, a_n\}} \subseteq F$ .

Suppose now that  $a$  is von Neumann regular but  $r(a) \notin \partial_e(F_1)$ , that is,  $r(a) \in F$  is not a complete tripotent, or equivalently,  $F_0(r(a)) \neq \{0\}$ . By Proposition 2(b) (cf. [26, Proposition 10]) we have  $E_2(r(a)) = F_2(r(a)) \subseteq F$ . We note that  $a$  also is von Neumann regular in  $E$ . Finally, it is known that for a von Neumann regular element  $a$  in a JB\*-triple  $E$  we have  $Q(a)(E) = Q(r(a))(E) = E_2(r(a))$



(cf. [31, comments after Lemma 3.2] or [9, p. 192]). This shows that  $Q(a)(E) = E_2(r(a)) = F_2(r(a)) \subseteq F$ .  $\square$

A (closed) subtriple  $I$  of a JB\*-triple  $E$  is said to be a *triple ideal* or imply an *ideal* of  $E$  if  $\{E, E, I\} + \{E, I, E\} \subseteq I$ . If we only have  $\{I, E, I\} \subseteq I$  we say that  $I$  is an *inner ideal* of  $E$ . Following standard notation, given an element  $a$  in  $E$ , we denote by  $E(a)$  the norm-closure of  $Q(a)(E) = \{a, E, a\}$  in  $E$ . It is known that  $E(a)$  is precisely the norm-closed inner ideal of  $E$  generated by  $a$  (cf. [6]).

As we commented above, let  $E_a$  the JB\*-subtriple of  $E$  generated by  $a$ . Clearly,  $a^{[3]} = Q(a)(a) \in E_a \subseteq E(a)$ , and hence  $E(a^{[3]}) \subseteq E(a)$ . By the Gelfand theory for JB\*-triples, there exist a subset  $L_a \subseteq [0, \|a\|]$ , with  $L_a$  compact and  $\|a\| \in L_a$ , and a triple isomorphism  $\Psi$  from  $E_a$  onto  $C_0(L_a)$ , satisfying  $\Psi(a)(t) = t$  ( $t \in L_a$ ) (cf. [29, Lemma 1.14]). By the Stone-Weierstrass theorem, we know that  $E_a = E_{a^{[3]}}$ . Therefore  $a \in E_{a^{[3]}} \subseteq E(a^{[3]})$ , which implies that  $E(a^{[3]}) \supseteq E(a)$ . Therefore  $E(a^{[3]}) = E(a)$ .

The previous Proposition 3 implies that, if  $F$  is a Čebyšev JB\*-subtriple of a JB\*-triple  $E$ , then for each non Brown-Pedersen quasi-invertible element  $a$  in  $F$ , the inner ideal of  $E$  generated by  $a$  is contained in  $F$ . We can actually prove a stronger conclusion.

**Corollary 4.** *Let  $F$  be a Čebyšev JB\*-subtriple of a JB\*-triple  $E$ . For each  $a$  in  $F$  which is not Brown-Pedersen quasi-invertible in  $F$  the inner ideal generated by  $a$  in  $E$  coincides with the inner ideal of  $F$  generated by  $a$ , that is,  $E(a) = F(a)$ .*

*Proof.* Let us observe that  $a$  is not Brown-Pedersen quasi-invertible in  $F$  if and only if  $a^{[3]}$  satisfies the same property. Proposition 3 shows that  $E(a) = \overline{Q(a)(E)} \subseteq F$ . That is,  $E(a)$  is an inner ideal of  $F$  which contains  $a$ , and hence  $F(a) \subseteq E(a)$ .

Now, we fix  $x \in E$ . The same Proposition 3 above implies that  $Q(a^{[3]})(x) = Q(a)Q(a)Q(a)(x) \subseteq Q(a)(F) \subseteq F(a)$ , and consequently,  $E(a) = E(a^{[3]}) = \overline{Q(a^{[3]})(E)} \subseteq F(a)$ .  $\square$

Let us observe that in Theorem 1 cases (a), (b) and (c), the JBW\*-subtriple  $F$  contains BP quasi-invertible elements. The remaining case (d) motivates us to consider the following partial result to determine the Čebyšev JB\*-subtriples of a general JB\*-triple.

**Corollary 5.** *Let  $F$  be a Čebyšev JB\*-subtriple of a JB\*-triple  $E$ . Suppose  $F$  contains no BP quasi-invertible elements (equivalently,  $\partial_e(F_1) = \emptyset$ ). Then  $F$  is an inner ideal of  $E$ .*

*Proof.* By hypothesis  $F$  contains not BP quasi-invertible elements, then Proposition 3 implies that  $\{a, E, a\} \subseteq F$  for every  $a \in F$ . A standard polarization argument shows that  $\{a, E, b\} \subseteq F$  for every  $a, b \in F$ , witnessing that  $F$  is an inner ideal of  $E$ .  $\square$

**Lemma 6.** *Let  $F$  be a JB\*-subtriple of a JB\*-triple  $E$ . Suppose  $F$  contains no BP quasi-invertible elements (equivalently,  $\partial_e(F_1) = \emptyset$ ). Then for each  $e \in \partial_e(E_1)$  we have  $\text{dist}(e, F) = 1$ . If  $F$  is a Čebyšev subspace of  $E$ , we have  $\pi_F(e) = 0$ , for every  $e$  as above.*

*Proof.* Suppose we can find  $x \in F$  satisfying  $\|e - x\| < 1$ . Then

$$\|e - P_2(e)(x)\| = \|P_2(e)(e - x)\| \leq \|e - x\| < 1.$$

Since  $e$  is the unit element of the JB\*-algebra  $E_2(e)$ , we deduce that  $P_2(e)(x)$  is an invertible element in  $E_2(e)$ . Lemma 2.2 in [25] implies that  $x$  is BP quasi-invertible in  $E$ , and hence BP quasi-invertible in  $F$ , which is impossible. The second statement is clear because  $\text{dist}(e, F) = 1 = \|e\|$ .  $\square$

The following technical lemma will play an useful role in subsequent results.

**Lemma 7.** *Let  $V$  be a closed Čebyšëv subspace of a Banach space  $X$ . Let  $S : X \rightarrow X$  be a surjective linear isometry satisfying  $S(V), S^{-1}(V) \subseteq V$ . Then  $S(\pi_V(x)) = \pi_V(S(x))$ , for every  $x \in X$ .*

*Proof.* We claim that

$$\text{dist}(x, V) = \text{dist}(S(x), V),$$

for every  $x \in X$ . Let us pick  $x \in X$ . Since  $S$  is a surjective linear isometry,  $\text{dist}(x, V) = \|x - \pi_V(x)\| = \|S(x) - S(\pi_V(x))\|$ , with  $S(\pi_V(x)) \in V$ . Therefore

$$\text{dist}(x, V) \geq \text{dist}(S(x), V).$$

Applying the same arguments to  $S^{-1}$ , we deduce that

$$\text{dist}(x, V) \geq \text{dist}(S(x), V) \geq \text{dist}(S^{-1}S(x), V) = \text{dist}(x, V).$$

$\square$

We recall that, given a tripotent  $e$  in a JB\*-triple  $E$  and  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$ , the mapping

$$S_\lambda : E \rightarrow E, S_\lambda = \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e)$$

is an isometric triple isomorphism (compare [20, Lemma 1.1]).

We establish now an strengthened version of Proposition 2.

**Proposition 8.** *Let  $F$  be a Čebyšëv JB\*-subtriple of a JB\*-triple  $E$ . Suppose that  $a$  is a non-zero element in  $F$ . Then  $\{a\}_E^\perp = \{x \in E : x \perp a\} \subseteq F$ .*

*Proof.* Arguing by contradiction, we suppose the existence of an element  $x \in \{a\}_E^\perp \setminus F$ . From the axioms in the definition of JB\*-triples, we know that for each  $t \in \mathbb{R}$ , the mapping  $S_t = \exp(itL(a, a)) : E \rightarrow E$  is a surjective linear isometry (triple isomorphism) with inverse  $S_t^{-1} = S_{-t}$ . Since  $a \in F$  and  $F$  is a JB\*-subtriple of  $E$ , we deduce that  $L(a, a)^n(F) \subseteq F$ , and hence  $S_t(F) \subseteq F$ , for every  $t \in \mathbb{R}$ .

Applying Lemma 7 it follows that  $S_t(\pi_F(x)) = \pi_F(S_t(x))$ , for every  $t \in \mathbb{R}$ . Having in mind that  $a \perp x$  it follows that  $L(a, a)^n(x) = 0$ , for every natural  $n$ , which shows that  $S_t(x) = x$  for every real  $t$ . Therefore

$$\pi_F(x) = \pi_F(x) + itL(a, a)(\pi_F(x)) + \sum_{n=2}^{\infty} \frac{i^n t^n}{n!} L(a, a)(\pi_F(x)).$$

Differentiating at  $t = 0$  we conclude that  $L(a, a)(\pi_F(x)) = 0$ , or equivalently  $a \perp \pi_F(x)$  (cf. [7, Lemma 1]).



We have proved that  $a \perp x, \pi_F(x)$ . Therefore  $\pi_F(x) + \mu a \in F$ , for every  $\mu \in \mathbb{C}$  and, by orthogonality,

$$0 < \text{dist}(x, F) = \|x - \pi_F(x)\| = \max\{\|x - \pi_F(x)\|, \|\mu a\|\} = \|x - \pi_F(x) - \mu a\|,$$

for every  $\mu \in \mathbb{C}$  with  $\|\mu a\| \leq \|x - \pi_F(x)\|$ , contradicting the uniqueness of the best approximation of  $x$  in  $F$ .  $\square$

We recall that a tripotent  $u$  in the bidual of a  $JB^*$ -triple  $E$  is said to be *open* when  $E_2^{**}(u) \cap E$  is weak\* dense in  $E_2^{**}(u)$  (see [14]). A tripotent  $e$  in  $E^{**}$  is said to be *compact- $G_\delta$*  (relative to  $E$ ) if there exists a norm one element  $a$  in  $E$  such that  $e$  coincides with  $s(a)$ , the support tripotent of  $a$  (see [14]). A tripotent  $e$  in  $E^{**}$  is said to be *compact* (relative to  $E$ ) if there exists a decreasing net  $(e_\lambda)$  of tripotents in  $E^{**}$  which are compact- $G_\delta$  with infimum  $e$ , or if  $e$  is zero.

Closed and bounded tripotents in  $E^{**}$  were introduced and studied in [16] and [17]. A tripotent  $e$  in  $E^{**}$  such that  $E_0^{**}(e) \cap E$  is weak\* dense in  $E_0^{**}(e)$  is called *closed* relative to  $E$ . When there exists a norm one element  $a$  in  $E$  such that  $a = e + P_0(e)(a)$ , the tripotent  $e$  is called *bounded* (relative to  $E$ ) (cf. [16]). Theorem 2.6 in [16] (see also [19, Theorem 3.2]) asserts that a tripotent  $e$  in  $E^{**}$  is compact if, and only if,  $e$  is closed and bounded.

**Corollary 9.** *Let  $F$  be a Čebyšev  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ . Let  $e$  be a tripotent in  $F^{**}$  satisfying that  $F_2^{**}(e) \cap F \neq \{0\}$ . Then  $\{e\}_E^\perp = \{x \in E : x \perp e\} = E \cap E_0^{**}(e) \subseteq F$ . Furthermore, if  $e$  is closed in  $E^{**}$  we also have  $E_0^{**}(e) = F_0^{**}(e)$ .*

*Proof.* By hypothesis, the set  $F \cap F_2^{**}(e)$  is non-zero, thus, there exists a non-zero element  $a \in F \cap F_2^{**}(e)$ . It is easy to check that  $\{e\}_E^\perp \subseteq \{a\}_E^\perp$ , and the latter is contained in  $F$  by Proposition 8.

We have already proved that  $\{e\}_E^\perp = E \cap E_0^{**}(e) \subseteq F$ , which implies that  $E \cap E_0^{**}(e) = F \cap F_0^{**}(e)$ . Since  $e$  is closed in  $E^{**}$ , we can assure that

$$E_0^{**}(e) = \overline{E \cap E_0^{**}(e)}^{\sigma(E^{**}, E^*)} = \overline{F \cap F_0^{**}(e)}^{\sigma(E^{**}, E^*)} \subseteq F_0^{**}(e) \subseteq E_0^{**}(e).$$

$\square$

**Corollary 10.** *Let  $F$  be a Čebyšev  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ . Let  $a$  be a non-zero element in  $F$  and let  $r(a)$  denote the range tripotent of  $a$  in  $F^{**}$ . Suppose that  $\{a\}_F^\perp \neq \{0\}$ . Then  $E_0^{**}(r(a)) = F_0^{**}(r(a))$ .*

*Proof.* We can assume that  $\|a\| = 1$ . Let  $F_a$  denote the  $JB^*$ -subtriple of  $F$  (or of  $E$ ) generated by  $a$ . We have already commented that there exists  $L_a \subseteq [0, 1]$ , with  $L_a$  compact and  $1 \in L_a$ , and a triple isomorphism  $\Psi$  from  $F_a$  onto  $C_0(L_a)$ , satisfying  $\Psi(a)(t) = t$  ( $t \in L_a$ ) (cf. [29, Lemma 1.14]).

Proposition 3 and Corollary 4 imply that  $E(a) = F(a)$ . Therefore  $E(a) = F(a)$  is an open  $JB^*$ -subtriple of  $E^{**}$  relative to  $E$  in the sense employed in [16, 18, 19]. Proposition 3.3 in [19] (or [16, Corollary 2.9]) implies that every compact tripotent in  $F(a)^{**}$  is compact in  $E^{**}$ . Let us take a compact tripotent  $e \in F(a)^{**}$  satisfying that  $e \leq r(a)$  and  $F_2^{**}(e) \cap F \neq \{0\}$  (we can consider, for example  $e = \chi_{[\delta, 1] \cap L_a}$  the characteristic function of the set  $[\delta, 1] \cap L_a$  in  $F_a$  and

$$y(t) := \begin{cases} 0, & \text{if } t \in [0, \delta]; \\ \text{affine}, & \text{if } t \in [\delta, 2\delta]; \\ 1, & \text{if } t \in [2\delta, 1]. \end{cases} \quad \text{in } F_a, y \in F_2^{**}(e) \cap F \text{ with } 0 < \delta < 2\delta < 1).$$

Since  $e$  is compact, and hence closed in  $E^{**}$  (cf. [16, Theorem 2.6]), Corollary 9 proves that  $E_0^{**}(e) = F_0^{**}(e)$ . Finally, it is easy to see that, since  $r(a) \geq e$ ,  $E_0^{**}(r(a)) \subseteq E_0^{**}(e) = F_0^{**}(e) \subseteq F^{**}$ , and hence  $E_0^{**}(r(a)) = F_0^{**}(r(a))$ .  $\square$

We turn now our focus to the Peirce one subspace associated with a range tripotent.

**Lemma 11.** *Let  $F$  be a Čebyšëv  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ . Suppose  $a_1, a_2, b_1, b_2$  are elements in  $F$  satisfying that  $a_1 + b_1$  and  $a_2 + b_2$  are not Brown-Pedersen quasi-invertible in  $F$ , and  $a_i \perp b_j$  for every  $i, j \in \{1, 2\}$ . Then  $L(b_1, b_2)L(a_1, a_2)(E) \subseteq F$ .*

*Proof.* Since  $a_1 + b_1$  and  $a_2 + b_2$  are not Brown-Pedersen quasi-invertible in  $F$ , Proposition 3 assures that

$$Q(a_j, b_j)(E) \subseteq Q(a_j + b_j)(E) + Q(a_j)(E) + Q(b_j)(E) \subseteq F, \quad (3.2)$$

for every  $j = 1, 2$ . The identity

$$Q(a_1, b_1)Q(a_2, b_2) = L(b_1, b_2)L(a_1, a_2)$$

can be easily deduced from the Jordan identity and the orthogonality of  $a_i$  and  $b_j$ . The last identity together with (3.2) prove the desired statement.  $\square$

We can establish now our first main result on Čebyšëv  $JB^*$ -subtriples of a general  $JB^*$ -triple.

**Theorem 12.** *Let  $F$  be a Čebyšëv  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ . Suppose  $F$  has rank greater or equal than three. Then  $E = F$ .*

*Proof.* Since  $F$  has rank greater or equal than three, we can find mutually orthogonal norm-one elements  $a, b, c$  in  $F$ . Let  $F_a, F_b$ , and  $F_c$  denote the  $JB^*$ -subtriples of  $F$  generated by  $a, b$ , and  $c$ , respectively. Since  $F$  is a  $JB^*$ -subtriple of  $E$ , they also coincide with the  $JB^*$ -subtriples of  $E$  generated by  $a, b$ , and  $c$  respectively. We observe that  $c$  is orthogonal to every element in  $F_a \oplus F_b$ . So, given  $a_1, a_2 \in F_a$  and  $b_1, b_2 \in F_b$ , we apply Lemma 11 to deduce that

$$L(a_1, a_2)L(b_1, b_2)(E) \subseteq F, \quad (3.3)$$

for every  $a_1, a_2 \in F_a$  and  $b_1, b_2 \in F_b$ .

Let us consider two bounded sequences  $(a_n) \subset F_a, (b_n) \subset F_b$  converging to  $r(a)$  and  $r(b)$  (the range tripotents of  $a$  and  $b$  in  $F^{**}$ ) in the strong\*-topology of  $F^{**}$ , or equivalently, in the strong\*-topology of  $E^{**}$  (cf. [4, Corollary]). Fix  $x \in E$ . It follows from (3.3) that  $L(a_n, a_n)L(b_n, b_n)(x) \in F$ , for every natural  $n$ . The joint strong\*-continuity of the triple product on bounded sets of  $E^{**}$  implies that  $L(r(a), r(a))L(r(b), r(b))(x) \in F^{**} \equiv \overline{F}^{\sigma(E^{**}, E^*)}$ , for every  $x \in E$ . The weak\*-density of  $E$  in  $E^{**}$  (cf. Goldstine's theorem) and the separate weak\*-continuity of the triple product of  $E^{**}$  assures that

$$L(r(a), r(a))L(r(b), r(b))(E^{**}) \subseteq F^{**}. \quad (3.4)$$

It is well known that  $L(r(a), r(a)) = P_2(r(a)) + \frac{1}{2}P_1(r(a))$ , and a similar identity holds for  $r(b)$ . Therefore (3.4) implies that

$$\left( P_2(r(a)) + \frac{1}{2}P_1(r(a)) \right) \left( P_2(r(b)) + \frac{1}{2}P_1(r(b)) \right) (E^{**}) \subseteq F^{**}. \quad (3.5)$$

It is well known that  $r(a) \perp r(b)$  implies

$$E_2^{**}(r(a) + r(b)) = E_2^{**}(r(a)) \oplus E_2^{**}(r(b)) \oplus (E_1^{**}(r(a)) \cap E_1^{**}(r(b)))$$

and

$$E_1^{**}(r(a) + r(b)) = (E_1^{**}(r(a)) \cap E_0^{**}(r(b))) \oplus (E_0^{**}(r(a)) \cap E_1^{**}(r(b))).$$

We thus deduce from (3.5) that

$$E_2^{**}(r(a) + r(b)) \subseteq F^{**}, \text{ and } E_1^{**}(r(a) + r(b)) \subseteq F^{**},$$

and hence

$$E_2^{**}(r(a) + r(b)) = F_2^{**}(r(a) + r(b)), \quad (3.6)$$

and

$$E_1^{**}(r(a) + r(b)) = F_1^{**}(r(a) + r(b)).$$

Since  $c \in \{a + b\}_F^\perp$ , we are in position to apply Corollary 10 to show that

$$E_0^{**}(r(a) + r(b)) = E_0^{**}(r(a + b)) = F_0^{**}(r(a + b)) = F_0^{**}(r(a) + r(b)). \quad (3.7)$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} E^{**} &= E_0^{**}(r(a + b)) \oplus E_1^{**}(r(a + b)) \oplus E_2^{**}(r(a + b)) \\ &= F_0^{**}(r(a + b)) \oplus F_1^{**}(r(a + b)) \oplus F_2^{**}(r(a + b)) = F^{**}. \end{aligned}$$

Finally, it is easy to check that, under these conditions,  $E = F$ , as desired.  $\square$

**Corollary 13.** *Let  $B$  be a Čebyšev  $C^*$ -subalgebra (respectively, a Čebyšev  $JB^*$ -subtriple) of a  $C^*$ -algebra  $A$ . Suppose  $B$  has rank greater or equal than three. Then  $A = B$ .  $\square$*

It is well known that every infinite dimensional  $C^*$ -algebra contains an infinite sequence of mutually orthogonal non-zero elements (cf. [28, Exercise 4.6.13]), that is every infinite dimensional  $C^*$ -algebra has infinite rank. Furthermore, Exercise 4.6.12 in [28] proves that every  $C^*$ -algebra with finite rank must be finite dimensional and hence unital (cf. [38, Theorem I.11.2]).

Let us observe that a non-zero  $C^*$ -algebra without unit must have infinite rank. Thus, the following result of Pedersen follows from Corollary 13.

**Corollary 14.** [34, Theorem 4] *Let  $B$  be a non-unital Čebyšev  $C^*$ -subalgebra of a  $C^*$ -algebra  $A$ . Then  $A = B$ .  $\square$*

It remains to study Čebyšev  $JB^*$ -subtriples of rank smaller or equal than two. In this case, the conclusion will follow from the main result in [26] and the studies about finite rank  $JB^*$ -triples developed in [5] and [2].

**Theorem 15.** *Let  $F$  be a non-zero Čebyšev  $JB^*$ -subtriple of a  $JB^*$ -triple  $E$ . Then exactly one of the following statements holds:*

- (a)  $F$  is a rank one  $JBW^*$ -triple with  $\dim(F) \geq 2$  (i.e. a complex Hilbert space regarded as a type 1 Cartan factor). Moreover,  $F$  may be a closed subspace of arbitrary dimension and  $E$  may have arbitrary rank;
- (b)  $F = \mathbb{C}e$ , where  $e$  is a complete tripotent in  $E$ ;
- (c)  $E$  and  $F$  are rank two  $JBW^*$ -triples, but  $F$  may have arbitrary dimension;
- (d)  $F$  has rank greater or equal than three and  $E = F$ .

*Proof.* If  $F$  has rank  $\geq 3$ , Theorem 12 implies that  $E = F$ . We may therefore assume that  $F$  has rank  $\leq 2$ . It follows from [5, Proposition 4.5 and comments at the beginning of §4] (see also [2, §3]) that  $F$  is reflexive. So,  $F$  is a reflexive  $JBW^*$ -triple of rank  $\leq 2$ .

We shall adapt next the arguments in the proof of [26, Theorem 13], we provide a simplified argument. Every  $JBW^*$ -triple admits an abundant collection of complete tripotents or extreme points of its closed unit ball (cf. [3, Lemma 4.1] and [32, Proposition 3.5] or [11, Theorem 3.2.3]). Thus, we can find a complete tripotent  $e$  in  $F$ . There are only two possibilities: either  $e$  is minimal in  $F$  or  $e$  has rank two in  $F$ .

When  $e$  is rank two in  $F$ . We can write  $e = e_1 + e_2$  with  $e_1, e_2$  mutually orthogonal minimal tripotents in  $F$ . Proposition 2 proves that  $E_2(e_j) = F_2(e_j) = \mathbb{C}e_j$ ,  $E_0(e_j) = F_0(e_j)$ , and  $E_0(e_1 + e_2) = F_0(e_1 + e_2) = \{0\}$ , which proves that  $e_1$  and  $e_2$  are minimal tripotents in  $E$ ,  $e$  is complete in  $E$ , and  $E$  is a rank-2  $JBW^*$ -triple.

We finally assume that  $e$  is minimal and complete in  $F$ . If  $\dim(F) = 1$ , then  $F = \mathbb{C}e$ , and we are in case (b), otherwise we are in case (a).  $\square$

It should be remarked here that Remark 7 in [26] provides an example of an infinite dimensional rank-one Čebyšev  $JB^*$ -subtriple of a  $JB^*$ -triple, while [26, Remark 13] gives an example of a rank-one Čebyšev  $JB^*$ -subtriple of a rank- $n$   $JBW^*$ -triple, where  $n$  is an arbitrary natural number.

In the setting of  $C^*$ -algebras, the following result of Pedersen follows directly from our Theorem 15.

**Corollary 16.** [34, Theorem 5] *If  $A$  is a unital  $C^*$ -algebra and  $B$  is a Čebyšev  $C^*$ -subalgebra of  $A$ , then  $A = B$ ,  $B = \mathbb{C}1$  or  $A = M_2(\mathbb{C})$  and  $B$  is the subalgebra of diagonal matrices.*  $\square$

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